Proportion in school mathematics textbooks: A comparative study

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Abstract. This paper analyses how proportion is introduced in mathematics textbooks for middle school students of Portugal, Spain, Brazil, and USA. The analysis focuses on the nature of the approach and on the cognitive demand, structure, and context of the tasks. The results show that textbooks mostly present tasks at an intermediate level of cognitive demand, with a closed structure, and in non-mathematical contexts. Furthermore, most textbooks organize the chapters in a similar way and follow a spiral approach. However, there are marked differences are in the way textbooks approach the conceptual and procedural aspects of proportion. The way the students are addressed also varies, ranging from a questioning and problem solving style, to an explaining/practicing style, supporting rather different kinds of student activity.

Key words. Textbooks. Task analysis. Proportion. Curriculum.

Introduction

Textbooks have a strong influence in mathematics teaching and learning. An aspect of concern for mathematics educators is how these educational instruments present key mathematical topics in relation to current curriculum orientations. This paper analyses how the topic of proportion – a central notion in the middle school mathematics curriculum – is presented in textbooks for 11-12 year old students of four different countries: Portugal, Brazil, Spain, and USA. We are particularly interested in how textbooks introduce and develop the notion of direct proportion, the tasks that they present to support students in consolidating and systematizing their knowledge, and noting other similarities and differences among textbooks. The focus of analysis is the nature of the tasks proposed to students given their key role in structuring students' learning activities (Christiansen & Walter, 1986).

Proportion in the mathematics curriculum and in mathematics education research

Proportion plays an important role in the curriculum of the four countries represented in this study. In Portugal, the syllabus and the national curriculum

documents (Ministério da Educação, 1991, 2001) recommend the development of the concept of direct proportion through real life situations from grade 6 on. In Brazil, the curriculum documents, *Parâmetros Curriculares Nacionais* (Ministério da Educação e Ciência, 1998), suggest that the preparation for the study of proportion must begin at grades 4 and 5, through rational numbers and percents, and, in grades 6 and 7, address the direct variation of proportional magnitudes. In Spain, the curriculum documents *Reais Decretos* (2003, 2004), suggest that direct proportional magnitudes to be studied at grade 7 (the first year of ESO, *enseñanza secundaria obligatoria*); however, percent is studied in grade 6 (last year of EP, *enseñanza primaria*). In the USA there is no official curriculum but we can have an idea of the trends in this country by looking at the documents issued by the NCTM (1989, 2000, 2006). These documents present proportion as a central notion that interrelate numbers and other topics in algebra and geometry. They support a preparation for understanding this concept since the first years of school through the study of patterns and regularities, common fractions, decimal fractions, and percents. NCTM (2006) indicates "Connecting ratio and rate to multiplication and division" as a curriculum focal point for grade 6 and "Developing an understanding of and applying proportionality, including similarity" as a focal point for grade 8.

There are many research studies dealing with teaching and learning of proportion. An important research program in this field was developed by Lesh, Post and Behr (1988). These mathematics educators regard proportion as the capstone of learning of numbers and operations and consider it as an essential basis for learning algebra and other topics. They suggest that working with multiplicative comparisons in the first years of school will lead the student to understand the equivalence of fractions, the conversion of measures, and how to deal with percents, ratio, and rate. They emphasize that proportional reasoning evolves through the development of local competencies based on contextualized knowledge. In their view, the student attains proportional reasoning when he or she is able to reason based on global relationships between rational expressions (fractions, quotients, ratios, and rates).

Spinillo (1992, 1996), from the standpoint of educational psychology, regards the development of proportional reasoning as progressing through the acquisition of multifaceted strategies by the student. She supports the idea that the development of proportional reasoning begins in the first years of school, as intuitive knowledge, giving particular attention to first order relations that are the basis for the establishment of

second order relations (connecting two or more first order relations). This author underlines the importance of using the notion of "half" as a powerful instructional resource. She also stresses the use of estimations to promote an effective differentiation between absolute and relative numbers and between first and second order relations.

The way textbooks present direct proportion was addressed in several studies (Cabrita, 1996; Ruggiero & Basso, 2003; Shield & Dole, 2002). Most textbooks present the formal notion of proportion as an equality between two ratios: $\frac{a}{b} = \frac{c}{d}$. However, in practical tasks, they often regard proportion as just a relationship between four numbers:

"If *a* is for *b* such as *c* is for *d*, we represent this as

a ---- b

c ---- d

and then we say that *a*, *b*, *c* and *d* are in proportion and *ad* = *bc*."

This is often called the "rule of three simple" (assuming that we know three of the values *a*, *b*, *c*, *d*, we can figure out the fourth), and then relationship *ad* = *bc* is usually named as the "cross product".

Previous studies identified two main approaches to deal with problems involving proportions. One uses the rule of three simple or cross product. The other approach emphasizes the "fundamental property of proportions", stating that "if $\frac{a}{b} = \frac{c}{d}$ is a proportion, then *ad* = *bc*". The main difference is that the "rule of three simple" or "cross product" is a relationship involving four numbers whereas the "fundamental property of proportions" involves the notions of ratio and equation.

Cabrita (1996) studied how seven Portuguese grade 7 textbooks revisit the notion of direct proportion previously taught at grade 6. She indicates that, to solve problems, some of the textbooks use the fundamental property of proportions and others use the rule of three simple. Shield and Dole (2002) analysed two chapters ("ratio and proportion" and "ratio and rates") of an Australian grade 7 textbook. The authors conclude that the most common method of solving proportion tasks is the "cross product" and "solving the proportion in order to *x*". They contend that textbooks do not make a proper distinction between ratio (part-part) and fraction and percent (part-whole). In their view, textbooks fail to promote the development of proportional

reasoning. Ruggiero and Basso (2003) analyze a Brazilian textbook that received a higher rank in the chapter in percent in a national textbook assessment. They conclude that the distribution of topics is horizontal, but the textbook fails to promote learning with understanding.

Mathematics tasks and connections

Mathematical tasks and connections play a central role in this study. Tasks are regarded as the proposals and challenges set by the teacher or the textbook (and in some cases even by the student) that constitutes a goal to attain (Christiansen & Walther, 1986; Ponte, 1995). The activity is what the students really do in terms of their thinking and actions. Therefore tasks are external and the activity is internal to the student. Skovsmose (2000) says that the teacher "invites" the student to get involved in the task. The way the student responds to such invitation is his or her activity. The NCTM (2000) also refers to tasks as something constructed by the teacher, to be proposed to the student. In another document, the NCTM (1991) stresses that tasks must support students in developing their ability to formulate and solve problems, communicate ideas, and establish mathematical connections.

The NCTM (2000) indicates that in designing a task one needs to take into account the content, the level of difficulty, the routine or non-routine nature, the complexity, and the degree of openness. Gimeno (1998) indicates that tasks need to be regarded with attention to the cognitive processes that they promote (memorization, comprehension, opinion, and discovery). In the PISA study (OCDE, 2004) there are three different kinds of tasks, according to the level of cognitive demand: reproduction, connection, and reflection. Also to differentiate the cognitive complexity of tasks, Smith and Stein (1998) and Stein and Smith (1998) speak of routine and non-routine tasks. They suggest that routine tasks include memorization tasks and tasks with no connections and non-routine tasks include tasks with connections and "doing mathematics". Ponte (2005) suggests a model that differentiates tasks according to the degree of openness and the degree of challenge, comprising exercises, explorations, problems, and investigations.

The tasks may differ in their context. Skovsmose (2000) proposes that contexts may be of reality, semi-reality, or purely mathematical. He contends that semi-reality situations may look like reality at first glance but, in fact, these situations often are

meaningless for the students. In recent years, context and connections become an increasingly important idea in school mathematics. For example, Pepin and Haggarty (2004) underline the importance of textbooks establishing connections within and outside mathematics. The NCTM (1991, 2000) indicates that mathematical connections constitute a basic standard in mathematics learning through all grade levels, from the primary to the secondary school. This includes both connections within mathematics and connections of mathematics with other fields. Portuguese curriculum documents (ME, 1991, 2001) indicate the need of interrelating teaching unities, developing transversal themes and using mathematics as tool for understanding and acting upon the real world. The PISA study (OCDE, 2004) puts a particular emphasis in the nature of the context underlying each task.

The nature of mathematics tasks has also been addressed in textbook research. For example, Li (2000) compared American and Chinese mathematics grade 7 textbooks, studying the kinds of tasks proposed in the topic of addition and subtraction of whole numbers. This author analysed five American textbooks and four Chinese textbooks using three criteria of analysis, concerning the mathematical dimension, the contextual dimension and performance requirements. She concluded that both American and Chinese textbooks mostly propose tasks that only require a single operation and are framed in a purely mathematical context. Also, Shield and Dole (2002) stress that tasks in Australian textbooks tend to be not contextualized in reality and to present few connections among mathematics topics as, for example, between proportion and geometry.

Methodology

As we indicated earlier, this paper aims to ascertain how textbooks introduce and develop the notion of direct proportion, to ascertain how they encourage students to consolidate and systematize their knowledge, and to identify the similarities and differences among textbooks. To achieve this aim we sought to answer to following more specific questions: (i) How is the textbook regarding its physical features, language and illustrations? (ii) What is the organization of the textbook and the structure of the chapters? (iii) How does the textbook consider ICT and the history of mathematics? (iv) At what point of the textbook is the concept of proportion introduced? (v) What concepts pave the way to the notion of proportion? (vi) How does

the textbook approach the notion of proportion? (vii) What is the level of cognitive demand of the tasks proposed in the textbook? (viii) What is the structure of the tasks proposed? (ix) In what contexts are tasks presented in the textbook? (x) What are the common features among the textbooks? and (xi) What are the significant differences among the textbooks?

This study is based in documental analysis and uses a technique of content analysis. First, we begun by choosing the topic. We selected proportion because, in most countries, it is a central topic of the curriculum for the 11-12 year old age range and allows for possible variations according to cultural traditions and emphasis. Second, we selected the countries to study. We were interested in comparing the Portuguese textbooks with the textbooks of countries with which Portugal has stronger cultural or educational connections. We chose: (i) Brazil, as it speaks the same language; (ii) Spain, as it speaks a similar language and has a common cultural root; and (iii) the USA, because it plays a leading role internationally in setting the mathematics curriculum. One must note that the NCTM (1989, 1991, 2000) documents have been quite influential in Portugal. On the other hand, curriculum documents and educational materials from Brazil and Spain are little known in our country and we are interested in knowing possible similarities and differences given the cultural affinities. Third, and finally, in these four countries we selected a textbook that would have a strong share of the market and could be readily available (Table 1).

Table 1 – Textbooks selected for the study.

Country	Textbook	Grade level
Portugal	Neves, M. A., Faria, L., & Azevedo, A. (2000). *Matemática* (6.º ano, 2.ª parte). Porto: Porto Editora.	6
Brazil	Lopes, A. J. (2000). *Matemática hoje é feita assim* (6.ª série). São Paulo: FTD.	6
Spain	Colera, J., & Gaztelu, I. (2005). *Matemáticas* (Educación Secundaria – 1). Madrid: Anaya.	7
USA	Maletsky, E. & Askey, R. (2007). *Math*. Orlando: Harcourt Brace & Company.	6

The four textbooks (6 chapters altogether) were analysed using an instrument consisting on a framework for global analysis and another framework for the analysis of tasks. The first framework takes into consideration in a first, more general level, the physical appearance, accessibility of language, nature and role of illustrations, structure of the chapters, use of information and communication technology (ICT) including calculators and role of history of mathematics., In a second level, specifically related to the topic of proportions the first framework addresses how the concept of proportion in presented, how its study is prepared, and what are the main notions and strategies for solving tasks that are emphasised.

The second framework is specifically related to the tasks and includes three points: cognitive demand, structure, and context. Following closely the PISA framework (OCDE, 2004), cognitive demand was classified as reproduction, connection, and reflection tasks. *Reproduction tasks* are routine tasks that involve the use of knowledge previously acquired and practiced, have a low degree of mathematical complexity and do not ask for arguments in the response. Their interpretation is straightforward and they do not require the use of different kinds of representations. Furthermore, reproduction tasks tend to be quite structured and to be presented in a simple and familiar context. *Connection tasks* require the establishment of relationships or chains of reasoning, procedures, or computations and ask for a certain level of interpretation. They may include a request for justification or a simple explanation. They tend to have a closed structure and to be presented in a familiar or almost familiar context. *Reflection tasks* are more complex and require a high level of interpretation and reasoning, ask for a solution that involves the coordination of several steps, and often demand a response with some written communication and argumentation. Their structure is often open or semi open and they are generally presented in less familiar situations.

The structure of the tasks was classified in three subcategories: closed, semi open and open (Ponte, 2005). In *closed* tasks the mathematical givens, goals, and conditions are clearly indicated. In *open* tasks the student needs to provide some further specification of givens, goals, and conditions. *Semi open* tasks lie in between these two.

Finally, we classified the context of tasks also according to a scheme similar to the PISA framework (OCDE, 2004). Such context may be intra mathematical or non mathematical. Non mathematical contexts include six subcategories: *daily life situations* involve personal situations or situations directly related to students' daily activities;

school situations refer to activities and processes that occur in the school context; *professional situations* correspond to a professional activity that students may be involved in the future; *life in society* concerns tasks related to life in community and in society*; other areas of knowledge* include tasks from subjects such as physics, geography, sports, language and so on; *imagination/fiction* concern tasks drawn in a fantasy world. Intra mathematical contexts include situations devoid of explicit non-mathematical elements; they are *between topics* if they refer explicitly to concepts taught in other chapters; otherwise, they are on the *same topic*.

Results

General aspects

We begin our analysis considering general aspects regarding the physical features of the textbooks, the language and illustrations that they present and the role given to ICT and to the history of mathematics. The analysis of the four countries shows that the textbooks address the notion of proportion at different grade levels – Portugal, Brazil and USA at grade 6, and Spain at grade 7. We note that the four textbooks have similar size formats (about an A4 or letter sheet). However, the American textbook has much more pages than the other three (quantificar!). The covers of the American, Brazilian, and Portuguese textbooks do not have any apparent connection with mathematics, whereas the cover of the Spanish textbook shows a strong relationship with the discipline.

All textbooks have a clear language, accessible to the students of these grade levels. The American, Brazilian and Spanish textbooks, albeit differently, show concern in focusing students' attention in new mathematics terms, whereas the Portuguese textbook does not. Furthermore, all textbooks are widely illustrated by pictures, drawings, or photographs that are, in one way or another, related with the issue or problem in question but often do not present relevant information for its solution.

The organization of the chapters follows a pattern that does not vary much from textbook to textbook. All the textbooks include introductory tasks, application tasks, and final consolidation tasks in different quantities and levels of complexity. All textbooks except the Brazilian present revision tasks in the beginning of a new chapter. In introducing new concepts the Portuguese and Spanish textbooks begin by presenting

a problematic situation and its solution, explain the concepts that follow from that example, present a synthesis, and, finally, propose a battery of tasks for practice. The American textbook follows the same pattern, but it stresses more the need for revision introducing at the beginning of each new section a small set of routine tasks to review concepts formerly studied (figure 1). The Brazilian textbook is noteworthy because it begins with problematic situations that are solved through a "conversation" among cartoon characters and only afterwards it proposes some tasks for the student to do. Furthermore, this is the only textbook that does not include an early formalization of the concepts.

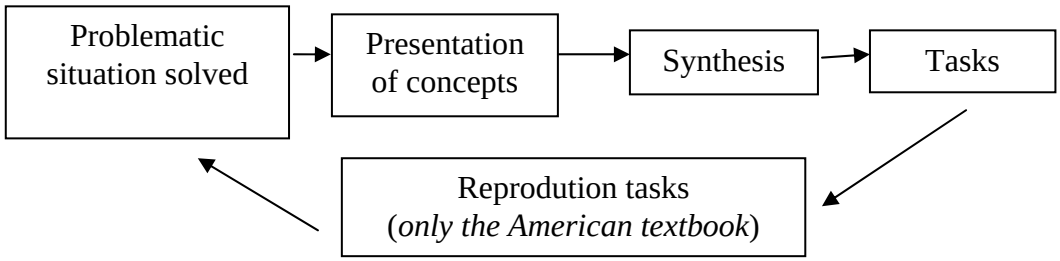


Figure 1 – Pattern of presenting new concepts in the textbooks studied.

All textbooks make few references to ICT or to the use of the calculator. The Spanish and American textbooks ask the pupil to use the calculator just a few times. The Portuguese and Brazilian textbooks do not include tasks that explicitly require the pupil to use the calculator.

Most textbooks do not have references to the history of mathematics in their chapters on proportion. The exception is the Brazilian grade 6 textbook, when introducing the notion of similar triangles. In a section entitled *Tales e a pirâmide* (Thales and the pyramid) there is a historical reference to Thales from Mileto and the way how, 2 700 years ago, this mathematician found the high of Quéops pyramid, using similarity of triangles (Figure 2). The textbook explaind that, assuming that the sun rays had the same inclination towards the ground, Thales considered two kinds of imaginary rectangular triangles: one formed by the stick, the sun rays and the shadow of the stick and another formed by the high of the pyramid, the sun rays and the shadow of the pyramid increased of half of the basis. As the internal angles were equal, Thales knew that the triangles were similar and, therefore, their sides were proportional.

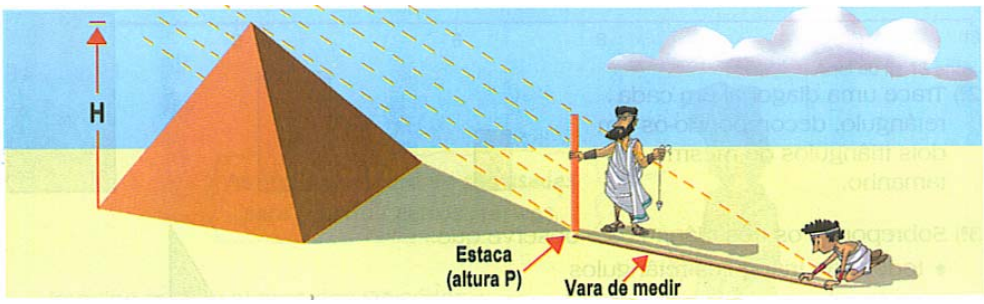


Figure 2 - Computation of the high of the pyramid of Queóps by Thales de Mileto (Brazilian textbook, grade 6, p. 262)

Proportion in the textbooks

Next we consider how the topic of direct proportion is included in the four textbooks and also how its study is prepared by previous chapters. For example, the Portuguese grade 6 textbook includes only one chapter on direct proportion. Before direct proportion, this textbook presents chapters about rational numbers and its operations and also on construction of triangles.

In the Spanish textbook of grade 7, direct proportion is the object of one chapter. Before this chapter it is studied natural, integer and decimal numbers, powers and square roots, divisibility, metric system. There is also a review of notions about rational numbers already taught in grade 6.

The Brazilian textbook for grade 6 presents proportion in two chapters, the second of which focuses on the study of geometric situations. Before introducting proportion, the previous chapters study the arithmetic mean, angles, operations with rational numbers, polygons and regularities, square root, negative numbers and equations.

The grade 6 American textbook deals with direct proportion in two chapters, the second of which focuses on percents. Before addressing proportion, it includes the study of numbers and operations, statistics and graphing, fraction concepts and operations, algebraic expressions, equations and patterns, geometry and plane figures, measurement in one and two dimensions and solid figures and measurement.

The textbooks introduce the concept of proportion in different ways. In the Portuguese textbook the study of concept is based on the previous study of rational numbers. The same happens with Spanish textbook that also builds on the previous study of percent. Meanwhile, in the Brazilian and American textbooks the study of proportion is based in the study of equations, patterns, and regularities, in line with the

NCTM's (2000) recommendations. It is clear that these four textbooks organize the mathematical content in rather different ways, preparing the study of this concept using different trajectories and carrying it out also differently.

Approach

There are also differences in the approach and content of the sections that made up the chapters of each of the textbooks analysed. As a starting point for the study of proportion, the Portuguese textbook presents the concept of ratio. Then, it addresses the notion of proportion and the fundamental propriety of proportions, and, afterwards, percents, scales, and direct proportion. The Spanish textbook for grade 7 approaches proportion using proportional series to discuss the relationship of proportional magnitudes, then presents direct and inverse proportion, and, finally, revisits the concept of percent in a deeper way. The Brazilian textbook grade 6, in a first chapter, presents ratio, percent, and change in proportional magnitudes, alluding to inverse proportional magnitudes. In a subsequent chapter, the textbook relates proportion with geometry, presenting the topics of scales, proportional rectangles, the fundamental property of proportions, and similar triangles. Finally, the American textbook introduces in chapter 24, the concepts of ratio, rate, proportion of similar figures and scales, and shows how to construct circular graphs and compute interest rates. Next, in chapter 25, it goes back to the concept of percent and enlarges it.

The approaches used by textbooks show three interesting contrasts. First, it is noteworthy that, in all countries except Spain, the textbooks begin the direct proportion chapter with the concept of ratio. The Spanish textbook for grade 7 begins it with "proportional series", addressing relationships between magnitudes. This is a rather unusual approach, that has a the advantage of leading to a close connection to the notion of function and to the use of tables to represent data but has the disadvantage of leaving aside the notion of ratio and the work with fractions. Second, in the Brazilian and American textbooks we note a spiral approach as they present the notion of direct proportion in one chapter and come back to it at least a second time. This does not happen in the Portuguese and Spanish textbooks that follow a linear approach, addressing this notion only once. Third, the fundamental property of proportions is stated in the Brazilian and Portuguese textbooks but not in the others. In contrast, the American and Spanish textbooks use the "cross product" (the American) or the rule of

three simple (the Spanish). As in the textbooks analysed by Cabrita (1996), these four textbooks approach suggest the rule of three simple/cross product as the main strategy to solve proportion tasks. For example, the Portuguese textbook presents the fundamental property of proportions as the first method to solve direct proportion tasks. However, it also presents the rule of three simple and the reduction to the unit as other possible forms for solving problems involving proportions (see Figure 3).

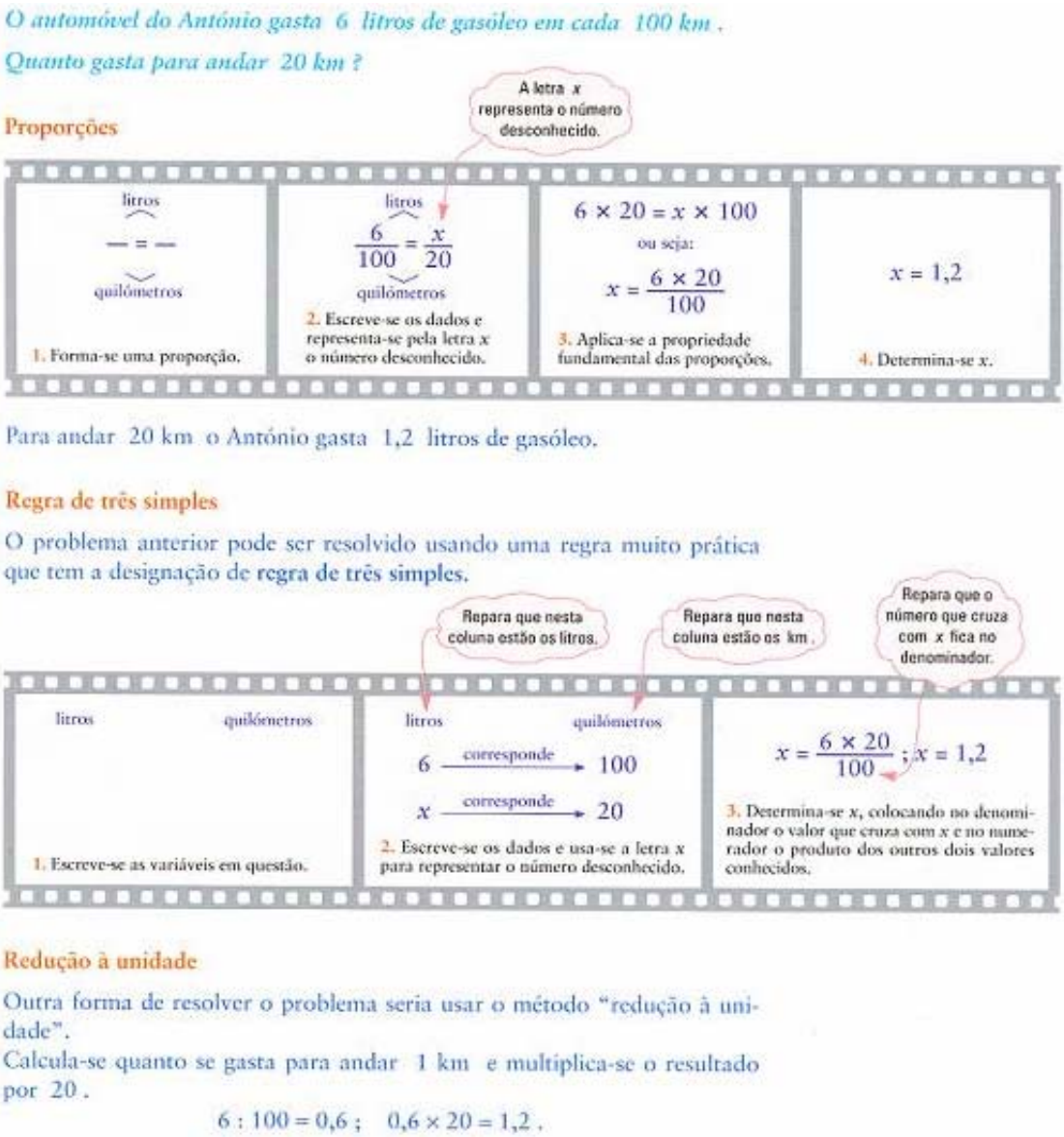


Figure 3 – Presentation of the fundamental property of proportions, cross product and reduction to unity (Portuguese textbook, grade 6, p. 50)

The Spanish textbook presents the "rule of three simple" followed by the cross product (see Figure 4). The textbook presents the schema of the rule without explaining it, then constructs a pair of fractions (representing ratios) that are intended as equivalent,

and finally encourages students to use the cross product. In another example it just moves from the “rule of three simple” to the cross product (see Figure 5).

PROBLEMA: Tres cajas de caramelos de café con leche cuestan 15 €. ¿Cuánto cuestan 5 cajas?

Si 3 cajas $\xrightarrow{\text{cuestan}}$ 15 € | $3 \rightarrow 15$

5 cajas $\xrightarrow{\text{costarán}}$? | $5 \rightarrow x$

Con los tres datos conocidos y el desconocido formamos una pareja de fracciones equivalentes:

$$\frac{3}{5} = \frac{15}{x} \rightarrow 3 \cdot x = 5 \cdot 15 \rightarrow 3 \cdot x = 75 \rightarrow x = 25 \text{ €}$$

Solución: Cinco cajas de caramelos cuestan 25 €.

Figure 4 – Rule of three simple in a solved example (Spanish textbook, grade 7, p. 159)

Consiste en formar una pareja de fracciones equivalentes con tres datos y la incógnita.

MAGNITUD 1		MAGNITUD 2
a	$\longrightarrow$	m
b	$\longrightarrow$	x

$$\frac{a}{b} = \frac{m}{x}$$

$$a \cdot x = b \cdot m \rightarrow x = \frac{b \cdot m}{a}$$

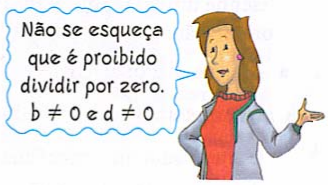
Figure 5 – Explanation of the rule of three simple (Spanish textbook, grade 7, p. 159)

The Brazilian textbook presents, by the end of the chapter, the fundamental property of proportions, in a very organized way and with all steps justified (see Figure 6).

Igualdades do tipo $\frac{a}{b} = \frac{c}{d}$ são chamadas de proporção.

Lê-se: "**a** está para **b**, assim como **c** está para **d**".

Os "elementos" **a**, **b**, **c** e **d** são chamados termos da proporção.



Como, na leitura da proporção $\frac{a}{b} = \frac{c}{d}$, iniciamos por **a** e terminamos por **d**, chamamos esses termos de extremos. Nessa leitura **b** e **c** vêm entre **a** e **d**, por isso **b** e **c** são termos conhecidos como meios.

Numa equação, se multiplicamos os dois membros pelo mesmo número, não nulo, a igualdade não se altera.

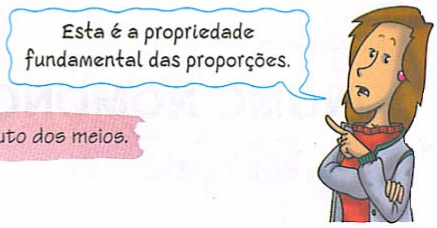
Para eliminar os denominadores, na igualdade $\frac{a}{b} = \frac{c}{d}$ multiplicamos o primeiro e o segundo membros por bd.

$$\frac{a \cdot \cancel{b}d}{\cancel{b}} = \frac{c \cdot b\cancel{d}}{\cancel{d}}$$

No primeiro membro os "bês" do numerador e do denominador se cancelam; no segundo membro os "dês" do numerador e do denominador também se cancelam.

O resultado é a igualdade ad = bc.

Se $\frac{a}{b} = \frac{c}{d}$, então $a \cdot d = b \cdot c$.



O produto dos extremos é igual ao produto dos meios.

Figure 6 –Fundamental property of proportions (Brazilian textbook, grade 6, p. 258)

The American textbook presents initially two ways to find proportions (see Figure 7). However, in the solved examples it places stronger emphasis in the cross product (see Figure 8).

$$\frac{2}{3} \stackrel{?}{=} \frac{9}{16} \qquad \frac{2}{3} \stackrel{?}{=} \frac{8}{12}$$

$$\frac{2 \times 16}{3 \times 16} \stackrel{?}{=} \frac{9 \times 3}{16 \times 3} \qquad 2 \times 12 \stackrel{?}{=} 8 \times 3 \leftarrow \text{cross products}$$

$$\frac{32}{48} \neq \frac{27}{48} \leftarrow \text{common denominators} \qquad 24 = 24$$

Figure 7 – Ways of finding proportions, including the cross product (American textbook, grade 6, p. 535)

Solve the proportion.

$$\frac{5}{2} = \frac{50}{a}$$

$\frac{5}{2} = \frac{50}{a}$	*Find the cross products.*
$5 \times a = 2 \times 50$	*Multiply.*
$5a = 100$	
$\frac{5a}{5} = \frac{100}{5}$	*Divide to solve the equation.*
$a = 20$	
$\frac{5}{2} = \frac{50}{a}$	*Check your solution*
$\frac{5}{2} \stackrel{?}{=} \frac{50}{20}$	*Replace a with 20. Find the cross products*
$5 \times 20 \stackrel{?}{=} 2 \times 50$	*Multiply.*
$100 = 100$ ✓	*The solution checks.*

Figure 8 – Solved example (American textbook, grade 6, p. 536)

Tasks

The tasks proposed in the textbooks vary in level of cognitive demand, structure and context.

Cognitive demand. Connection tasks predominate in the textbooks of the four countries, followed in second place by reproduction tasks (Table 1). The American textbook has the higher level of reflection tasks and has also the higher level of reproduction tasks. In the other three textbooks connection tasks clearly dominate (about 2/3 of the tasks proposed).

The cognitive demand of the tasks proposed varies between a minimum level of reproduction, an intermediate level of connection, to achieve a maximum level of reflection. In all textbooks analysed connection tasks predominate. In the Spanish and Brazilian textbooks such tasks represent 68% of the total, in the Portuguese textbook about 62% and in the American textbook only 47%. In a second place, textbooks present reproduction tasks. The American textbook stands up by the large percent of this kind of tasks. Reflection tasks are little represented in all textbooks. They have a stronger presence in the American (12%) textbook, followed by the Brazilian, Portuguese and Spanish.

Table 1 – Cognitive demand, structure and context of tasks presented in textbooks (%).

		Portugal	Spain	Brazil	USA
Cognitive	Reproduction	34	29	22	41

demand	Connection	62	68	68	47
	Reflection	4	3	10	12
Structure	Open	0	0	3	1
	Semi open	1	0	9	9
	Closed	99	100	89	90
Context	Intra mathematical	35	26	34	83
	Same topic	(26)	(26)	(21)	(66)
	Between topics	(9)	(0)	(22)	(17)
	Non mathematical	65	69	66	17
	Daily life situations	(5)	(2)	(3)	(2)
	School situations	(12)	(1)	(5)	(1)
	Professional situations	(13)	(11)	(7)	(0)
	Life in society	(34)	(58)	(32)	(8)
	Other areas of knowledge	(1)	(2)	(9)	(5)
	Imagination/fiction	(0)	(0)	(2)	(0)

Structure. The overwhelming majority of the tasks have a closed structure. Open tasks can only be identified in the Brazilian and American textbooks. Semi-open tasks could be identified in these textbooks and also in the Portuguese one, although at much lower level.

All the textbooks analysed present few tasks with an open or semi open structure. Such tasks do not exist in the Spanish textbooks. The textbook thats stand up because it offers tasks with a more open structure is the Brazilian, with 9% of semi open tasks and 3% of open tasks. The American textbook, that presents a high level of closed tasks (90%), proposes 9% of semi open tasks and 1% of open tasks. The Portuguese textbook is very similar in this regard to the Spanish, with no open tasks and only 1% of semi open tasks.

Context. In most textbooks analysed, the majority of tasks are framed in terms of non mathematical contexts. The exception is the American textbook, in which 83% of the tasks have intra mathematical contexts. The other textbooks contain little more than 30% of tasks relative to intra mathematical contexts, within the same chapter or related to other chapters. From these tasks framed in intra mathematical contexts, those that concern the exploration of concepts within the same mathematical theme are in larger number, especially in the American textbook (66% of the tasks are of this kind). This textbook presents a very low percent of non mathematical tasks (17%), when compared

with the remaining textbooks in which most tasks (near 70%) are related to students' daily activities, school situations, professional situations, life in society, other areas of knowledge and imagination. The most prominent subcategory of non mathematical context is life in society. The trend to stress mathematical contexts by American textbooks was already noted in the study by Li (2000)., and a similar result was reported by Shield and Dole (2002) regarding Australian textbooks.

Common and different aspects

As we have seen, the textbooks analyzed share several common features. The kinds and localization of the tasks is very similar. All textbooks have introductory, application, and revision tasks at the end of each section and again at the end of the chapter. The distribution of the cognitive demand of tasks is similar (with emphasis in connections) and the structure is also similar (with emphasis in closed tasks). Furthermore, all textbooks make little references to ICT or to the use of the calculator.

We also note some differences among the textbooks. The most striking difference is the strong emphasis of the American textbook in intra mathematical contexts whereas the textbooks of the other three countries clearly emphasize non-mathematical contexts. The American textbook is also noteworthy by its emphasis on revision tasks, not only at the end of sections and chapters, but also at the beginning of sections. A textbook with so few references to non mathematical contexts and with so a high stress on revision is certainly addressed to teachers using a different sort of pedagogy and to students with quite different learning culture.

Another important difference concerns the linear or spiral approach. The Portuguese and Spanish textbooks concentrates in a single chapter complementary mathematical themes, seeking to do a comprehensive discussion, adopting a linear conception of curriculum. In contrast, the Brazilian and American textbooks approach different concepts at different stages of the textbook or in different school years, presenting therefore a spiral, non-linear organization.

There is another important difference in the way the textbooks approach the mathematical notions and procedures. In one case (Portugal) the study proportion is based in the study of rational numbers, in another (Spain) based rational numbers and percents and (iii) in still another cases in the study of rational numbers, equations and patterns. In Portugal, Brazil and USA the notion of ratio plays in important role, but it

does not appear in Spain, that uses the notion of "proportional series". Only in two countries there is mention to the "fundamental property of proportions".

Finally, we stress differences in the teaching methodology. The most deviant case is the textbook from Brazil. This textbook is based in solving problematic situations that allows the systematization of concepts based in discussions about the issues presented. In contrast, in the methodology adopted in the Portuguese, Spanish and American textbooks is the role of the student is to understand the explanations and examples solved, to solve next a battery of proposed tasks, most of reproduction and connection. The way the students are addressed also varies in the textbooks, ranging from a questioning and problem solving style (Brazilian), to an explaining/ practicing style (Portugal, Spain, and USA). These two different styles may support rather different kinds of student activity.

Discussion

Having described several features of the way textbooks present proportion, one is led to ask what may be the reasons for such similarities and differences. One strong factor that influences textbooks is certainly the national curriculum documents. The Brazilian textbook is noteworthy as it emphasizes non mathematics contexts, presents a relatively higher quantity of cognitively demanding tasks and, most especially, follows an exploratory and informal approach, maintaining an extended dialogue with pupils before any formalizations. These features are well aligned with the orientations of the Brazilian curriculum documents *Parâmetros curriculares nacionais*. The Spanish textbook, in contrast, provides quick moves toward formalization and insists in tasks with low level of cognitive demand. It presents proportion without using the notion of ratio, but when it comes to teaching how to solve proportion tasks it suggests the use of the rule of three simple. This is well in line with the *Reais Decretos*, the Spanish official documents that stress a procedural and formal approach to mathematics. These textbooks provide two examples of strong alignment of textbooks and curriculum documents.

The American and Portuguese textbooks studied represent a different situation. In the American textbook there are features consonant with the NCTM (2000) *Principles and standards*, such as the emphasis on connections, some attention to cognitively demanding tasks, the emphasis in patterns and regularities, but there are features that depart from the orientations of this document such as the emphasis in intra

mathematical contexts and reproduction tasks. The high number of pages and tasks of this textbook suggest that teachers are expected to devise a specific trajectory within it for their students to follow. Also in the Portuguese textbook there are features strongly related to the curriculum documents, such as the use of non mathematical tasks or the lack of emphasis on connections, but others that are not in line with these documents, as the fact that if follows a linear and not a spiral approach. These two textbooks provide examples of weak alignment with curriculum documents.

The reasons for strong or weak alignment deserve some attention. First, they may relate to editorial options and with the pressure for achieving high level sales. The concern to adjust to taste of a wide audience of teachers is quite visible in the American, Spanish and Portuguese textbooks, whereas the Brazilian one decided to run the risk of adopting a more innovative style.

Second, strong or weak alignment may have to do with how the textbook is meant to be used by students. In Portugal and Spain the textbook is brought by families, is property of the student, and one of its important roles is to be used as study resource at home. Therefore, textbooks travel between home and the school and must not be very heavy. In the USA, in contrast, the textbooks are property of the school and are loaned to each student who must return it at the end of the year. Therefore there is no problem that textbooks have extra material that allows teachers more freedom to devise teaching trajectories with them.

And third, pedagogical traditions regarding mathematics education at each country may also constitute a strong influence in textbooks. This is quite apparent in the Spanish textbook. It follows the national documents by presenting proportion through the study of proportional series. However, instead of using this to connect to the notion of function and its representations (tables and graphs), it goes on teaching students about how to using the rule of three. The same happens with the Portuguese textbook that presents the fundamental property of proportions but goes on teaching the use of that rule to solve proportional tasks.

Conclusion

Teachers have to consider many issues regarding textbooks, both in process of analysis for adoption and for classroom use. Does the textbook include a variety of tasks in terms of cognitive demand, structure and context? Does it textbook promote a proper

use of ICT and makes reference to elements of the history of mathematics? The concepts and procedures presented in each chapter are related to the material of the previous chapters? The students are offered with tasks to do and stimulated to reflect on their work or mostly presented with rules to practice?

This paper shows how the study of mathematics textbooks may use instruments of analysis specifically oriented towards the teaching of this discipline. In this case, the focus was in the teaching approach and in the tasks proposed, what become a productive perspective of analysis from the point of view of curriculum orientations. Using this methodology, we found notable similarities but also striking differences in the textbooks analyzed in several respects. It may be that within each country we would found also great variations among textbooks. However, in some cases we see some connection between the curriculum documents that stress an innovative point of view and the nature of the approach and activity proposed in the textbooks. In other cases we also see how textbooks adjust to and reinforce conservative elements of curriculum documents and teaching practices. For future studies, it would be interesting to know how different kinds of textbooks are used in practice by students and teachers. Much importantly, it would be interesting to know in what measure the learning opportunities provided by innovative textbooks gets enacted in classroom practice and, most especially, in student learning.

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